§ 3.7 Extreme Values

#1. Suppose (1.1) is a critical pt of a fct $f \in C^2$.

In each case, what can you say about f ?

(a) $f_{xx}(1,1) = 4$, $f_{xy}(1,1) = 1$, $f_{yy}(1,1) = 2$

(b) $f_{xx}(1,1) = 4$, $f_{xy}(1,1) = 3$, $f_{yy}(1,1) = 2$

sol of (a)

$$\Delta = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 6 > 0 \quad \text{and} \quad f_{xx}(1,1) = 4 > 0$$

∴ (1,1) occurs a local minimum pt of f ※

sol (b)

$$\Delta = \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} = -1 < 0$$

⇒ f has a saddle pt at $(1,1)$ ※

3 – 14. Find local Max, minimum values and saddle pt

#3. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$

sol: 1° $\begin{cases} f_x = -2 - 2x \overset{?}{=} 0 \\ f_y = 4 - 8y \overset{?}{=} 0 \end{cases}$ ⇒ $x = -1$, $y = \frac{1}{2}$, c.p $(-1, \frac{1}{2})$

2° $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$

$$\Delta = \begin{vmatrix} -2 & 0 \\ 0 & -8 \end{vmatrix} = 16 > 0 \quad \text{and} \quad -2 < 0$$

∴ $(-1, \frac{1}{2})$ is a local Max and $f(-1, \frac{1}{2}) = 11$ is a l.M. value. ※

#7. $f(x,y) = (1 + xy)(x + y) = x + x^2y + y + xy^2$

sol: 1° $\begin{cases} f_x = 1 + 2xy + y^2 \overset{?}{=} 0 & —① \\ f_y = x^2 + 1 + 2xy \overset{?}{=} 0 & —② \end{cases}$ ① – ② ⇒ $y^2 = x^2$ ⇒ $y = \pm x$

当 $y = x$ 代入 ① $3x^2 + 1 = 0$ 无解

当 $y = -x$ 代入 ① $x^2 = 1$ ⇒ $x = \pm 1$

∴ 有2个 c.p $(1, -1)$ 及 $(-1, 1)$

2° $f_{xx} = 2y$, $f_{xy} = 2x+2y$, $f_{yy} = 2x$

$$\mathrm{Hess}(f) = \begin{bmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{bmatrix} = 2\begin{bmatrix} y & x+y \\ x+y & x \end{bmatrix}$$

For $(1,-1)$: $\bar{\Delta} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 < 0$

$\therefore$ $(1,-1)$ is a saddle pt. ※

For $(-1,1)$: $\bar{\Delta} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 < 0$

$\therefore$ $(-1,1)$ is also a saddle pt. ※

#9. $f(x,y) = e^x \cos y$

sol: 1° $\begin{cases} f_x = e^x \cos y \overset{\text{令}}{=} 0 \\ f_y = -e^x \sin y = 0 \end{cases}$ $e^x > 0, \forall x \in \mathbb{R}$

$\Rightarrow \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}$ No such y. Thus there are no c.p. ※

#11. $f(x,y) = x \sin y$

sol: 1° $\begin{cases} f_x = \sin y \overset{\text{令}}{=} 0 & -① \\ f_y = x \cos y = 0 & -② \end{cases}$

by ① $\Rightarrow$ $y = n\pi$, $n \in \mathbb{Z}$ 代入 ② $\Rightarrow x = 0$

Thus all c.p. are $(0, n\pi)$, $n \in \mathbb{Z}$.

2° $f_{xx} = 0$, $f_{xy} = \cos y$, $f_{yy} = -x \sin y$

$$\mathrm{Hess}(f) = \begin{bmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{bmatrix} = -\cos^2 y$$

$\Delta(0, n\pi) = -\cos^2 n\pi = -1 < 0$, so each c.p is a saddle pt. ※

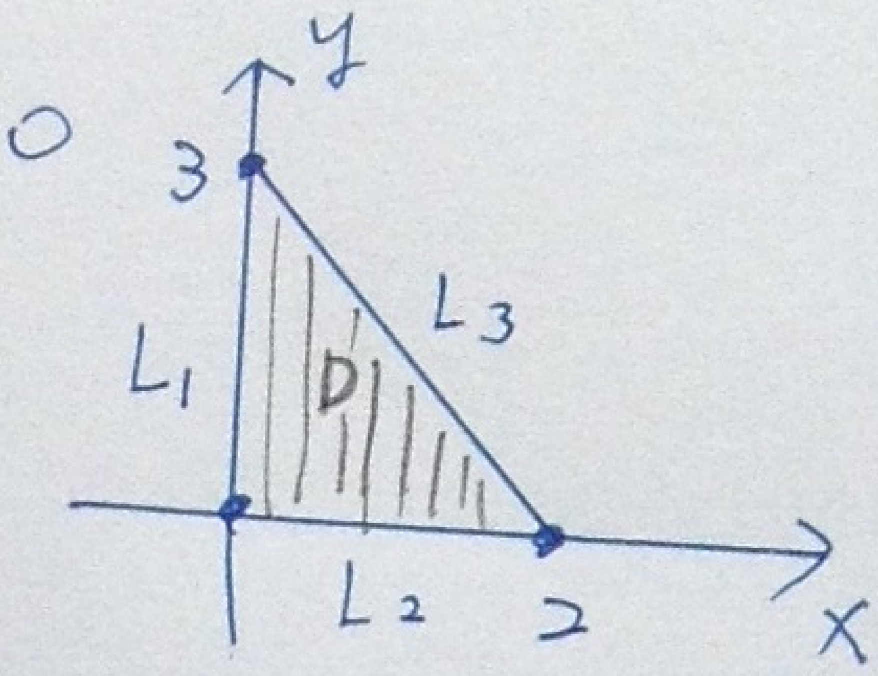
§3.7.

23–28. Find $\underset{D}{\operatorname{Max}} f$ and $\underset{D}{\min} f$

#23. $f(x,y) = 1 + 4x - 5y$, D is closed triangular region with vertices $(0,0)$, $(2,0)$ and $(0,3)$

sol: 1° 內部 c.p : $f_x = 4 \neq 0$

No c.p.



2° bdy

L_1: $x=0$, $0 \le y \le 3$, $f(0,y) = 1 - 5y \in [-14, 0]$

當 $x=0, y=3$

L_2: $y=0$, $0 \le x \le 2$, $f(x,0) = 4x + 1 \in [1, 9]$

當 $x=2, y=0$

L_3: $y = -\frac{3}{2}x + 3$, $f(x, -\frac{3}{2}x + 3) = \frac{23}{2}x - 14$

$0 \le x \le 2$, $-14 \le \frac{23}{2}x - 14 \le 9$

當 $x=0, y=3$　　當 $x=2, y=0$

3° 比較內部及 bdy 中最大者為 9

〃　　小者為 -14

$\underset{D}{\operatorname{Max}} f = 9$, $\underset{D}{\min} f = -14$.

#25. $f(x,y) = x^2 + y^2 + x^2y + 4$, $D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$

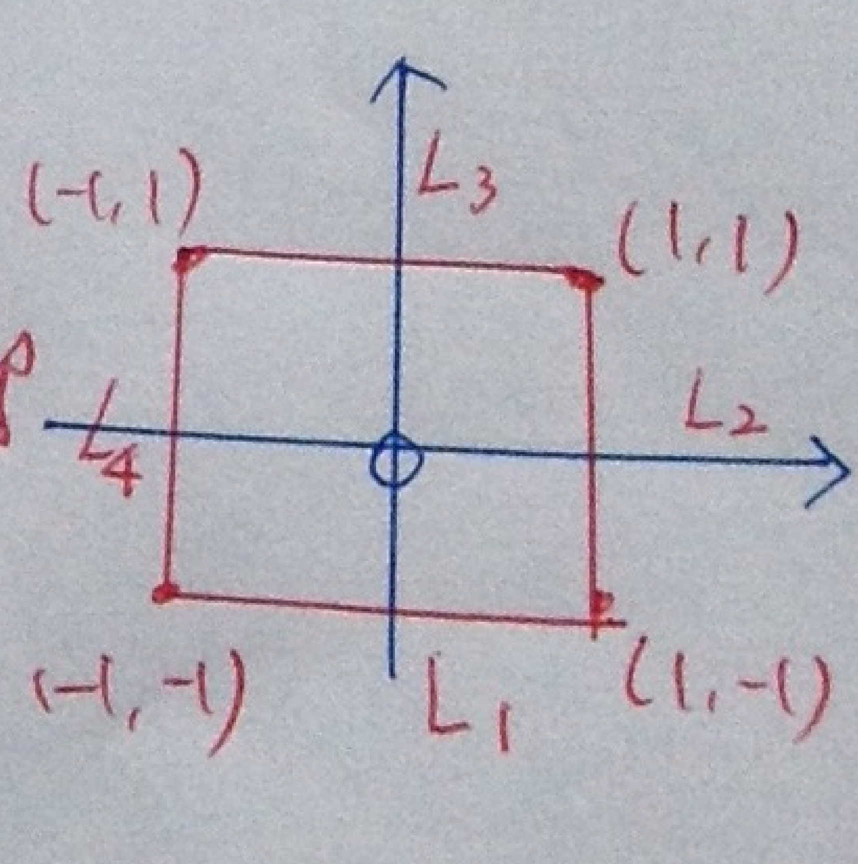
sol: 1° 內部: $\begin{cases} f_x = 2x + 2xy \overset{\text{令}}{=} 0 & \text{—①} \\ f_y = 2y + x^2 \overset{\text{令}}{=} 0 & \text{—②} \end{cases}$ ⇒ $x(y+1) = 0$ ⇒ $x = 0$ or $y = -1$

$x=0$ 代入② ⇒ $y=0$

$y=-1$ 代入② ⇒ $x = \pm\sqrt{2}$ 不在內部

∴ 內部只有 1 個 c.p. $(0,0)$

$f(0,0) = 4$



2° bdy

On L_1: $y=-1$, $-1 \le x \le 1$, $f(x,-1) = 5$

On L_3: $y=1$, $-1 \le x \le 1$, $f(x,1) = 2x^2 + 5 \in [5, 7]$

On L_2 : $x=1$, $-1\le y\le 1$, $f(1,y)=y^2+y+5=(y+\frac{1}{2})^2+\frac{19}{4}\in[\frac{19}{4},7]$

On L_4 : $x=-1$, $-1\le y\le 1$, $f(-1,y)=y^2+y+5=(y+\frac{1}{2})^2+\frac{19}{4}\in[\frac{19}{4},7]$

3° $\max_D f=7$; $\min_D f=4$ ※

#33. Find the pts on the cone $z^2=x^2+y^2$ that are closest to the pt $(4,2,0)$.

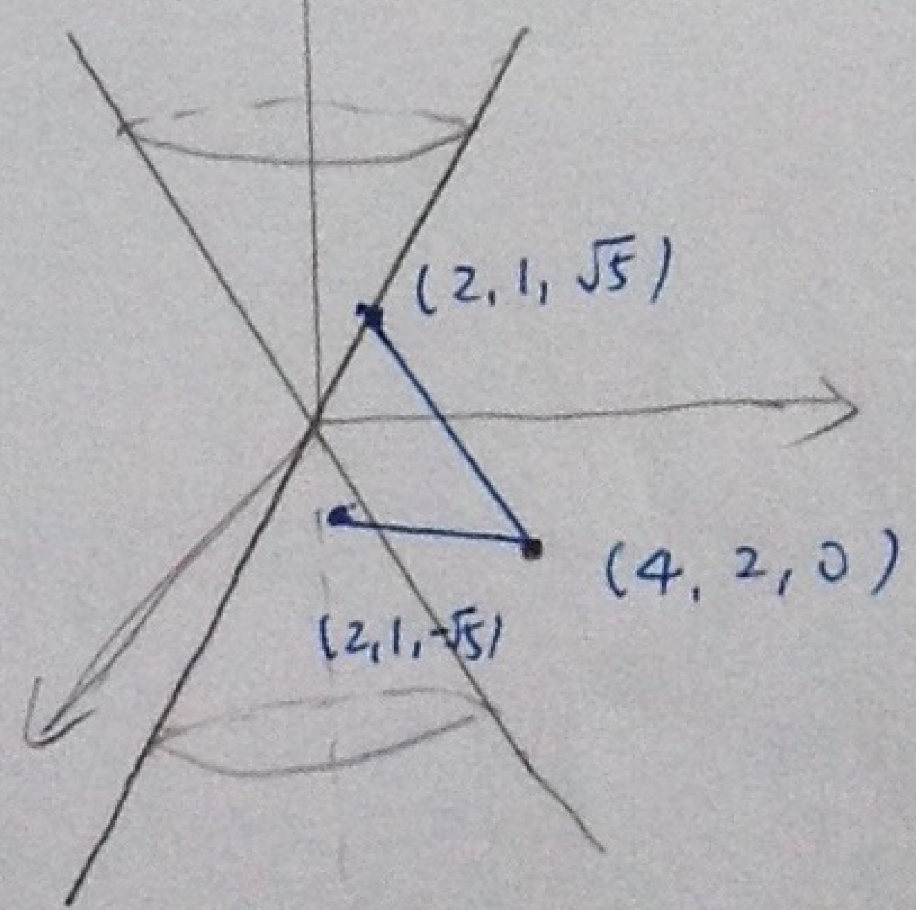
sol: Consider $d^2=(x-4)^2+(y-2)^2+z^2$ with $z^2=x^2+y^2$

$$=(x-4)^2+(y-2)^2+x^2+y^2$$

Let $f(x,y)=(x-4)^2+(y-2)^2+x^2+y^2$ and find $\min_{\mathbb{R}^2} f$

$$\begin{cases} f_x=2(x-4)+2x \overset{\text{令}}{=} 0 \\ f_y=2(y-2)+2y \overset{\text{令}}{=} 0 \end{cases} \Rightarrow x=2,\ y=1$$

Thus the only c.p $(2,1)$. Since absolute minimum exists which must occur at a c.p $x=2$, $y=1 \Rightarrow z=\pm\sqrt{5}$

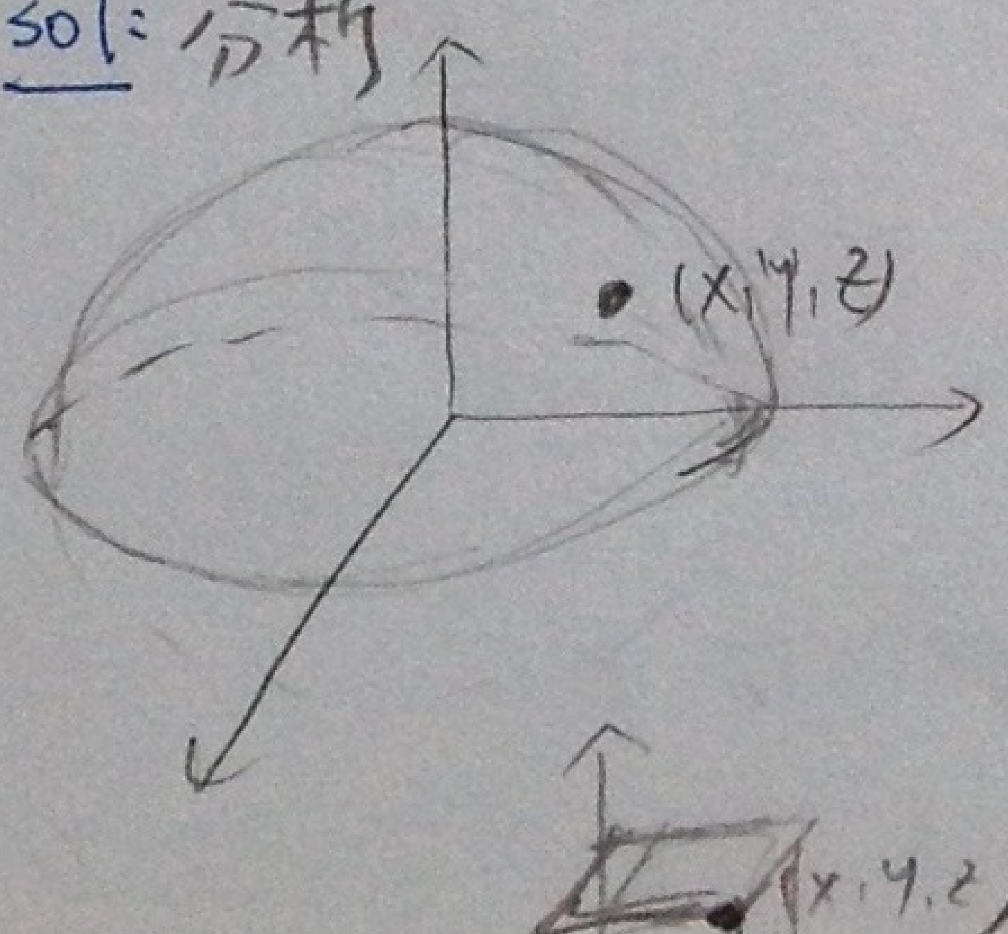


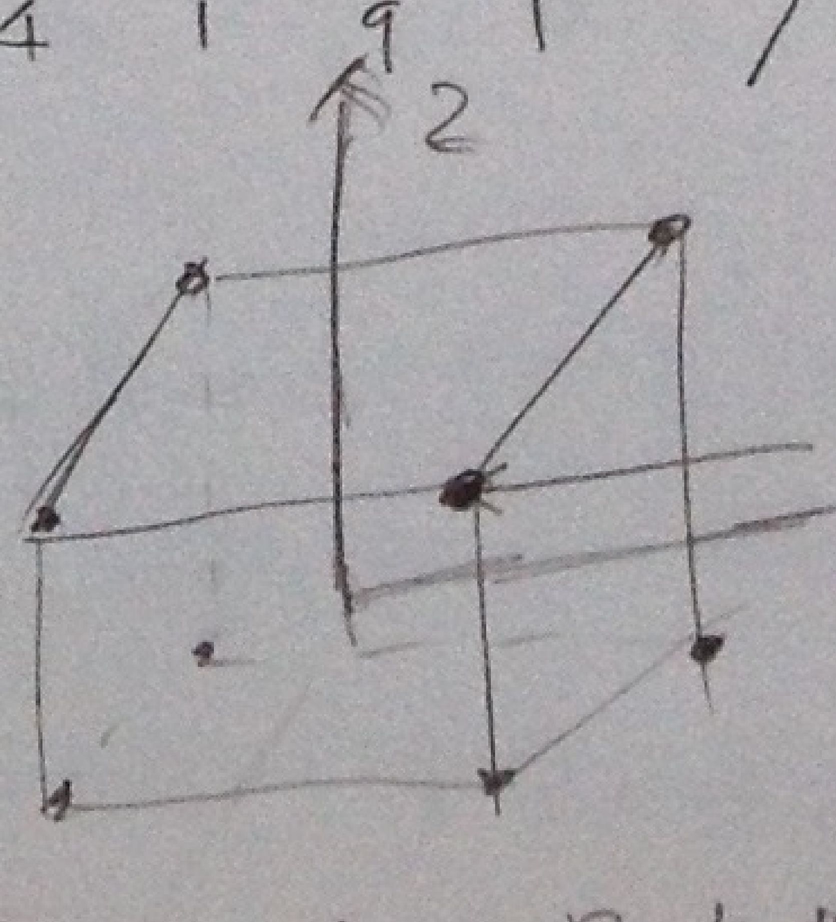
#37 Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid.

$$9x^2+36y^2+4z^2=36$$

sol: 分析

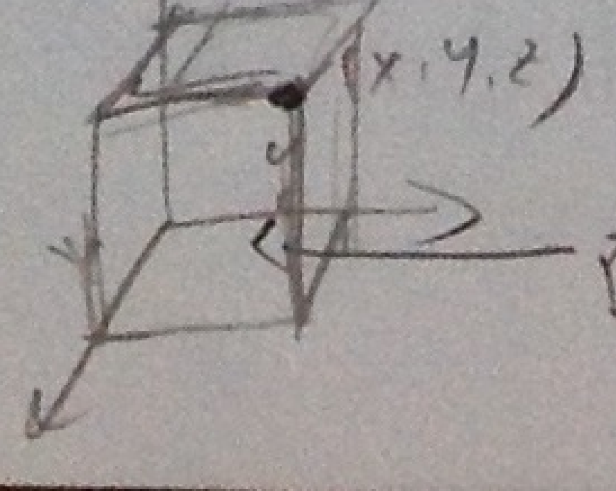
$$\frac{x^2}{4}+\frac{y^2}{1}+\frac{z^2}{9}=1$$





第一卦限

(x,y,z) lies in first octant



在第一卦限 是占其 $\frac{1}{8}$ 且 $V=xyz$

sol: $V = xyz$ with $z^2 = \frac{1}{4}(36-9x^2-36y^2)$ and $x, y, z > 0$

Let $f(x,y) = xy\frac{1}{2}\sqrt{36-9x^2-36y^2}$

$\underset{(x,y)在第一象限}{\text{Max}} f = ?$ $\quad f_x = \dfrac{y\sqrt{36-9x^2-36y^2}}{2} + \dfrac{-9x^2y}{2\sqrt{36-9x^2-36y^2}}$

$= \dfrac{36y-18x^2y-36y^3}{2\sqrt{36-9x^2-36y^2}}$

and $f_y = \dfrac{36x-9x^3-72xy^2}{2\sqrt{36-9x^2-36y^2}}$

Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$, $f_x = 0 \Rightarrow \because y \neq 0$, $x^2+2y^2-2=0$ —①

$f_y = 0 \because x \neq 0 \Rightarrow$, $x^2+8y^2-4=0$ —②

②−① 得 $y^2 = \frac{1}{3} \Rightarrow y = \frac{1}{\sqrt{3}}$ ($y<0$不合) 代入① $x = \frac{2}{\sqrt{3}}$

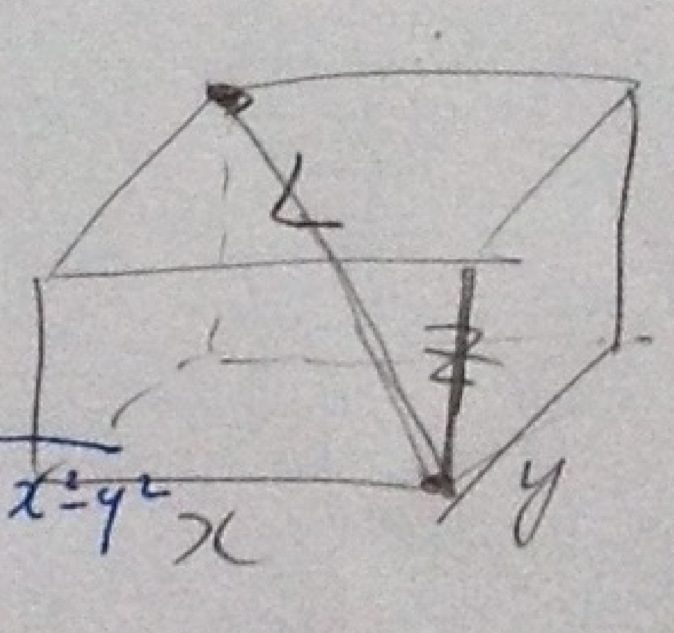
Hence $z = \frac{1}{2}\sqrt{36-12-12} = \sqrt{3}$. The fact that this gives a max volume follows from the geometry.

$\therefore \text{Max } V = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)(\sqrt{3}) = \frac{16}{\sqrt{3}}$

#45. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

sol: Let x, y, z be the dimensions of the rectangular box.

$\Rightarrow V = xyz$ and $L = \sqrt{x^2+y^2+z^2}$



We have $z = \sqrt{L^2-x^2-y^2}$

Substituting, we have $V(x,y) = xy\sqrt{L^2-x^2-y^2}$

$\underset{x,y>0}{\text{Max}} V(x,y) = ?$ $\quad V_x = y\sqrt{L^2-x^2-y^2} + \dfrac{-x^2y}{\sqrt{L^2-x^2-y^2}} = \dfrac{y(L^2-2x^2-y^2)}{\sqrt{L^2-x^2-y^2}}$

$V_y = \dfrac{x(L^2-2y^2-x^2)}{\sqrt{L^2-x^2-y^2}}$. By $V_x = 0$ and $y \neq 0 \Rightarrow 2x^2+y^2 = L^2$ —①

By $V_y = 0$ and $x \neq 0 \Rightarrow 2y^2+x^2 = L^2$ —②

①+② $x^2+y^2 = \frac{2}{3}L^2$ —③ 由①−③ 得 $x^2 = \frac{1}{3}L^2 \Rightarrow x = \frac{L}{\sqrt{3}}$

$y = \frac{L}{\sqrt{3}} \Rightarrow z = \frac{L}{\sqrt{3}}$ Thus, max $V = \frac{L^3}{3\sqrt{3}}$ #